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A MIXED MODE ROCK FRACTURE MODEL FOR THE PREDICTION OF CRACK PATH

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SUMMARY

Crack propagation in rocks is simulated by using a displacement substitution method based on a mixed mode fracture criterion. The main advantage of this model is that it can distinguish between mode I and mode II stress intensity factors simultaneously. A typical finite element program is used to compute displacements adjacent to the crack tip. The maximum circumferential tensile stress is adopted as a 'yield surface' for the calculation of the load factor in each crack increment. Pure mode I and mixed mode examples have been analysed to validate the capability of the model. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: rock mechanics; crack propagation; mixed mode

1. INTRODUCTION

In metals, crack propagation is mainly due to tensile loads. Nevertheless, in certain cases, the combination of shear and tensile loads are considered. In rocks, the applied loads are mainly compressive. If there are flaws, holes or cracks then regions of tensile stresses are evolved.

An important parameter in characterising various cracked bodies in the *stress intensity factor*, K, or alternatively the energy release rate, G, which depends on the crack shape and size as well as the loading condition and its magnitude. It is presumed that each material has a threshold value for K, known as the critical stress intensity factor, K_c, or *fracture toughness*. It is also assumed that fracture toughness is a material property, beyond which value the crack will propagate. Three independent modes of fracture have been recognized.

mode I: opening mode,
mode II: sliding mode,
mode III: tearing mode.

At the crack tip, there is a stress concentration which causes a singularity, so in order to distinguish between different cracked bodies, the strength of the singularity is calculated via the stress intensity factor, K. Three independent factors are known to exist, namely K_{I}, K_{II} and K_{III}.

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In practical problems we may often encounter a combination of modes. For example if a crack is inclined, as shown in Figure 1, the situation is referred to as a *mixed mode* for the case of *Linear Elastic Fracture Mechanics* (LEFM) which often prevails in the rock fracture mechanics context. Indeed, for rocks the usual loading results in compressive and shear loads which leads to a combination of modes I and mode II.

2. MODELLING OF CRACK PROPAGATION

For the numerical modelling of a single crack propagation it is necessary to evaluate the stress intensity factors. For a given geometry and load configuration it is usually feasible to calculate K, which depends strongly on the initial crack length. If the computed K (for a given mode) equals or exceeds the critical one, K_c, then the crack will propagate, and will continue to do so as long as the required energy for the crack propagation is available. Neither engineering rock mechanics or fracture mechanics claim that a fracture once initiated is unstable. Depending on the precise situation, the crack may stop after an increase of length even if the load is increasing up to a certain level and hence increasing the total (potential) energy. The crack length may increase gradually with an increase of the applied load, until unstable conditions are reached, when (local) rupture would occur. Therefore, crack propagation may be stable in the load control sense. Having reached a maximum, the stress intensity factors will decrease with increasing crack length and in order to propagate the crack further then more energy is required. The simultaneous propagation of several cracks is a probable feasibility. Crack propagation, in general, is not in its own plane because of the mixed mode condition when crack initiation may occur with $K_I < K_{I_c}$.

Sometimes, the engineer is interested in creating more controlled cracks, e.g. in the case of geothermal reservoirs and rock cut-off. On the other hand, crack propagation may be disastrous, e.g. for tunnelling and slope stability.

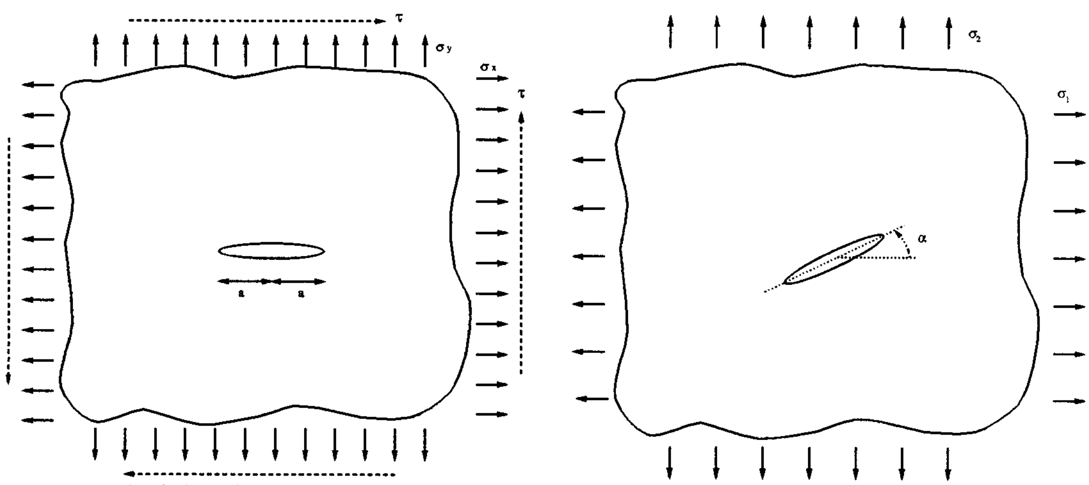

Figure 1. Combined stresses in a mixed mode condition

Each numerical model is expected to be able to evaluate the stress intensity factors versus new crack lengths, hence they are updated at each step. It is convenient to use a unit load vector which is parallel to the applied load, since in LEFM stress intensity factors are linearly proportional to the applied stresses. Then it is easy to find the necessary load to produce the desired stress intensity factors for crack propagation. According to the given mixed mode fracture initiation and propagation theory, the new crack propagation angle could then be calculated. In order to proceed two options are available, either the prediction of the required load to drive a crack a given increment, or alternatively, the crack increment evaluation for an applied load.

The main idea when selecting a model for simulation was to use algorithms which needed least modification to available finite element programs. Therefore, the method can be matched with almost all general purpose finite element programs.

3. STRESS INTENSITY FACTOR EVALUATION

During each crack length increment, an accurate evaluation of mode I and mode II stress intensity factors is inevitable for the prediction of load and angle change necessary for the corresponding length configuration for the next successive iteration. Since the singularity at the crack tip node is always of the same order, then if the singular stress (or strain) is multiplied by $r^{1/2}$ the resultant will be finite. For example, for mode I the stress intensity factor, K_{I}, can be defined as

$$K_{\mathrm{I}} = \lim_{r \to 0} r^{1/2} \sigma_y(a + r, 0) \tag{1}$$

where σ_y stands for tensile stress, a is the half crack length and r is the distance of an arbitrary point from the crack tip. The origin of the co-ordinate system used is at the crack centre.

Hence, according to the geometry of the body, the load configuration and the intensity, K_{I} is different for various cracked bodies. Different methods have been used for the evaluation of such stress intensity factors, e.g. Hybrid direct method; correlation methods; energy methods. Various techniques have been explained by Hellen in Chapter 5 of Reference[1]. In the hybrid direct method, K is computed directly by assuming it is a nodal unknown along with the displacement components. Even though this method seems to be accurate enough, especially in comparison with the other two categories, its main disadvantage is the need for a special element stiffness formulation which is not available in most standard finite element programs. For this reason, this method has often been overlooked. In the second category, the total potential energy, equal to the strain energy in LEFM, is computed for the original crack configuration and then for a very small advance of the crack it is necessary to evaluate the rate of energy released at the crack tip. Since there is a relationship between energy release rate and stress intensity factor, it is feasible to evaluate K at the crack tip. Some disadvantages are also associated with this method, mainly, the calculation must be run at least twice in order to compute the energy release rate, which is expensive computationally and should be avoided if possible. For mixed mode cases, the total energy is a function of the stress intensity factors of all the involved modes. Thus, the separation of individual stress intensity factors is another difficulty which needs additional computational effort. For this programming reason, correlation methods, sometimes called substitution methods, appear to be a good alternative for this purpose.

4. MODELLING OF SINGULARITIES

By removing the mid-side nodes of eight-noded elements to the quarter points, the Jacobian matrix would be singular at the crack tip node.

The above changes do not necessitate any changes to the shape functions used, therefore no modifications are required for the stiffness matrix and hence general purpose finite element programs are applicable which is another advantage. Indeed, as Barsoum[2,3] has concluded, the convergence characteristics of these special elements are not changed. It was also proven that if the eight-noded isoparametric quadrilateral elements adjacent to the crack tip are degenerated to triangle-like elements with three nodes coalescing at the crack tip node, without any changes to the eight shape functions, then better results are achieved. It should be mentioned again that only those elements which are common at the crack tip node must be changed to the degenerated quarter point elements.

Because of the sharp increase in gradients of the stress and strain fields near a crack tip, primitive finite element simulators had to adopt rather fine meshes, at least near the crack tip, in order to cope with the singularities. Later, by using higher-order elements, e.g. isoparametric elements, the same order of accuracy was achieved with relatively coarse meshes. However, it was perceived that for more efficient numerical modelling, it was advantageous to use special kinds of elements at the crack tip to cope with the $r^{-1/2}$ singularity, where r is the distance of the point under consideration from the crack tip.

5. THE MAXIMUM TENSILE STRESS THEORY, $\sigma_{\theta_{\max}}$

The stress field near a crack tip, expressed in a polar co-ordinate system, has been given in various references.[4,5] In this theory, proposed by Erdogan and Sih,[6] the maximum circumferential tensile stress, $\sigma_{\theta_{\max}}$, governs the crack initiation and propagation. The theory states that

(a) Crack propagation will initiate in a radial direction at the crack tip.
(b) The plane of propagation is normal to the direction of maximum tensile stress.
(c) Crack propagation occurs if $\sigma_{\theta_{\max}}$ reaches a critical value which is material dependent.

In other words,

$$\cos\theta_0/2\left[\frac{K_{\rm I}}{K_{{\rm I}_c}}\cos^2\theta_0/2-\frac{3}{2}\frac{K_{\rm II}}{K_{{\rm I}_c}}\sin\theta_0\right]=1 \tag{2}$$

and

$$\cos\theta_0/2\,[K_{\rm I}\sin\theta_0+K_{\rm II}(3\cos\theta_0-1)]=0 \tag{3}$$

The above equations form a fracture initiation locus. To check whether or not a crack propagates, the $K_{\rm I}$ and $K_{\rm II}$ factors, corresponding to the present geometry and given load configuration, are substituted into equation (3) to find the direction of probable crack propagation, θ_0. Then, these three parameters, i.e. $K_{\rm I}$, $K_{\rm II}$ and θ_0 are substituted into the left-hand side of equation (2). If the left-hand side is less than unity, then the crack is stable, otherwise it will propagate. Indeed, if the point $K_{\rm I}$, $K_{\rm II}$, plotted in the corresponding plane, is inside the fracture loci, nothing happens until an increase in load takes place and hence an increasing value for $K_{\rm I}$ and $K_{\rm II}$. Alternatively, if the plotted point is either on or outside the locus, the crack would propagate in the direction defined by θ_0, obtained from equation (3). According to the given condition, the crack may stop after propagating a certain distance in the determined direction, θ_0, which means that the energy for driving the crack has vanished, so the applied load must be increased if additional crack propagation is expected. Alternatively, it is quite possible that the previously applied load is the maximum sustainable load for the created crack configuration and hence (local) rupture would occur at this stage.

6. CRACK INCREMENT PREDICTIONS

At this point, the method of computation of stress intensity factors, as well as using the given crack initiation and propagation theory for predicting the possibility of propagation and its angle change, are described. The next step is to select one of the following options:

(a) Considering a load change and then computing the corresponding crack length increment.
(b) Considering a crack length increment and evaluating the subsequent corresponding load change to cause this increment.

In the present study the latter has been chosen. For each step a given crack length increment is considered, which may be the same for all of the steps or just for some of the steps. Some different situations are discussed in order to have a better insight into the matter.

6.1. Case 1

In the first case, the equivalent stress intensity factor, K_{eq}, is increasing monotonically with the crack length increment. Here, an equivalent stress intensity factor implies the combination of stress intensity factors for the mixed mode condition according to the given theory. It may also be in a normalized form, where Figure 2 depicts this case.

Generally speaking, K is proportional to the square root of the crack length, a, hence for a constant applied load, K_{eq} has a quadratic shape with reference to the crack length. Assuming K_{I_c}, at the value shown in Figure 2, implies that for the initial available crack lengths less than a, the crack would not propagate. For crack lengths greater than a, these would propagate in an

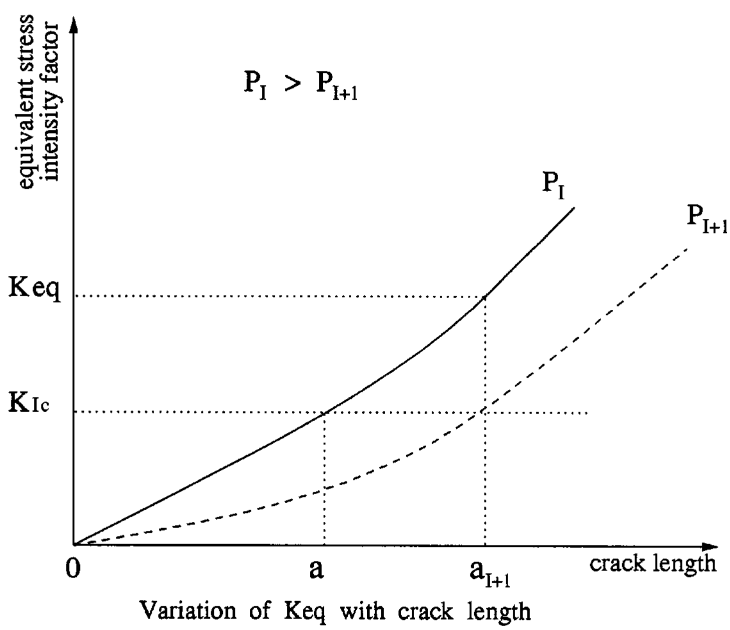

Figure 2. Variation of K_{eq} with crack length for the first case

unstable manner, unless the load level is decreased. The main idea behind crack propagation is that while there is sufficient energy to overcome the crack resistance force, the crack propagates. In other words, as long as the (equivalent) stress intensity factor is greater than, or equal to that of the critical one, the crack continues to propagate.

In this case, the cracked body can sustain less loads with increasing crack length. To compute the decreased load, one can consider the new crack length $a_{i+1} = a_i + \Delta a$ with the same load level P_i to obtain

$$K_{i+1} = fP_i\sqrt{a_{i+1}} \tag{4}$$

in which f depends on the geometry and the mixed mode theory in use. For the decreased load level, $P_{i+1} = P_i - \Delta P$, with the same geometry, the critical stress intensity factor, K_{I_c}, would be

$$K_{I_c} = fP_{i+1}\sqrt{a_{i+1}} \tag{5}$$

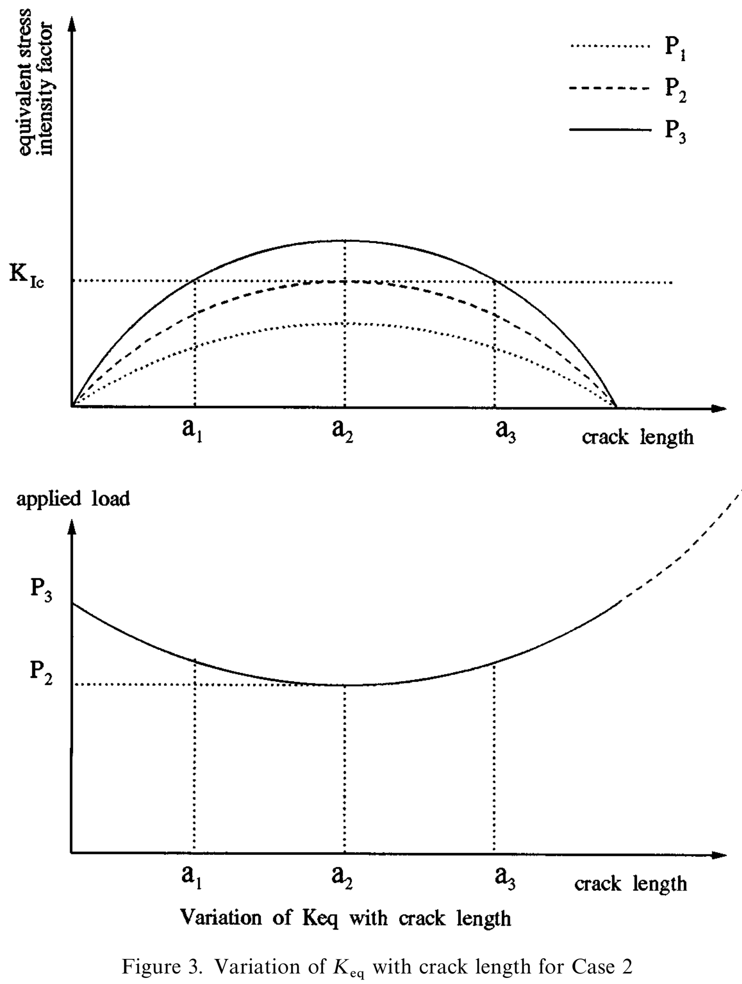

Figure 3. Variation of K_{eq} with crack length for Case 2

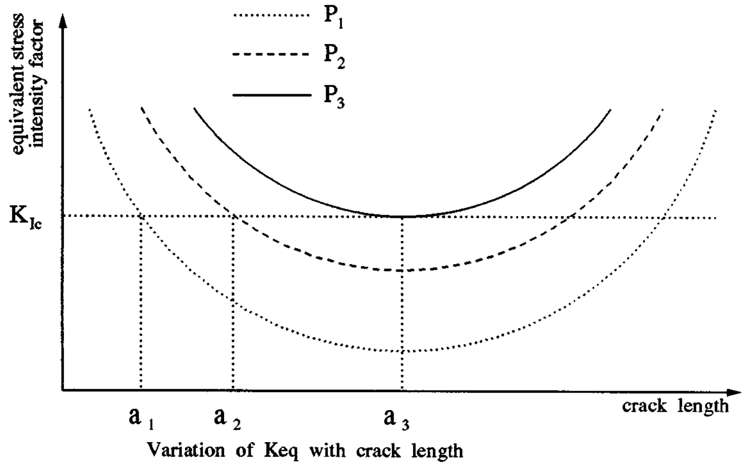

Figure 4. Variation of K_{eq} with crack length for Case 3

Table I. The results obtained for the first example series of pure mode I

Case	No. of nodes	Crack length	K_I	K_{II}	Angle change	Reference[9]
A	502	0·25	2659	149·8	− 6·41	2301
B	506 ×	0·25	2460	18·3	− 0·85	2301
C	516 ×	0·50	2957	114·1	− 4·40	3058

It is obvious that P_{i+1}, the maximum sustainable load for this case, is less than P_i. Using equations (4) and (5), we have

$$P_{i+1} = \frac{K_{I_c}}{K_{i+1}} P_i \tag{6}$$

6.2. Case 2

The second case to be discussed, depicts the situation where the equivalent stress intensity factor increases at first, reaches a maximum and then decreases, as shown in Figure 3. Three different load levels have been assumed.

Again, with reference to the given K_{I_c}, for load level P_1, no crack propagation would occur, whatever the crack length. In other words, P_1 is in the safe range. For load level P_2 the critical crack length is a_2, with the theoretical crack increment length $\Delta a = 0$. Indeed, the load level P_2 is a transient situation. After increasing the load, according to the level of load, there is always a critical crack length, a_1, which decreases with increasing load. Yet it is possible to choose a crack length increment which will finally attain the value a_2 by using equation (6), which yields a decreased load level associated with the crack increment length. The maximum sustainable load associated with a_2 is P_2. From this crack length, propagation of the crack would occur with

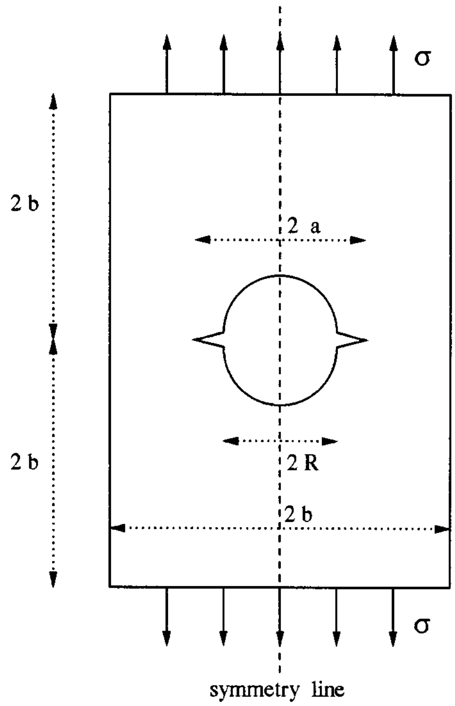

Figure 5. Two cracks at a circular hole

increasing the load value up to P_3, until the corresponding crack length, a_3, is reached. Figure 3 shows the variation of load, between P_3 and P_2, associated with crack length a_1 and a_2 respectively. Indeed, decreasing the load from P_3 and P_2 and then increasing again to P_3 is associated with a crack length varying from a_1 to a_2 and then to a_3.

6.3. *Case 3*

For the last case, in contrast to the previous one, the equivalent stress intensity factor decreases at first until reaching a minimum and then increases with respect to the crack length increase, as shown in Figure 4.

The lowest load level shown, P_1, associated with the critical crack length a_1, causes an increase in length at the available crack length. For the new geometry with the same load level, the equivalent stress intensity factor would decrease, therefore, to drive the crack it is necessary to increase the load level. In other words, the energy for creating new fractured surfaces has been depleted and under this load level no additional crack propagation would occur, i.e. a stable crack growth occurs.

Similarly, at the load level P_2, the critical crack length would be a_2 and yet the crack growth would be stable. Alternatively, for each arbitrary crack length a_i, between a_1 and a_3, there is an associated load level P_i to drive it in a stable manner. Finally, for the load level P_3, the crack

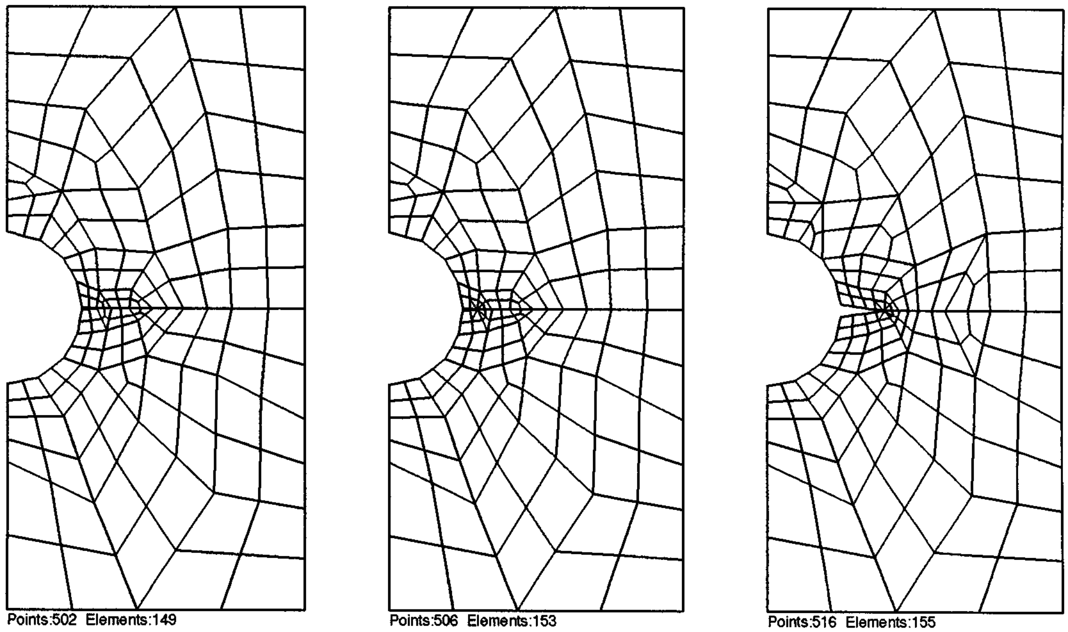

Figure 6. Various meshes used for the first example series, cases A, B and C, respectively

growth would be unstable since the normalised K_{eq} would always be greater than, or equal to, unity, i.e. global rupture would occur.

Again, in this case, equation (6) is applicable. Indeed, the aforementioned algorithm is a general one for all cases provided the crack length increment is reasonably chosen in order to minimise the resultant errors.

7. NUMERICAL RESULTS

In this section some examples will be presented to show the performance of the modified program in which PLASCON has been used as a general purpose finite element program. More details regarding PLASCON have been given by Lewis and Schrefler.[7] The example consist of pure mode I and mixed mode cases. In most cases, the degenerated triangle-like elements have been utilised in order to obtain more accurate results. The use of this kind of element has been shown in the associated tables with a $\times$ sign in front of the number of nodes in the related column.

7.1. Pure Mode I

As the first example two cracks at a circular hole in a rectangular sheet have been considered, as shown in Figure 5. Considering the symmetry, only half of the domain is solved. The applied tensile stress is normal to the axis of the cracks. Various crack lengths and openings have been considered in order to compare the differences in the results obtained.

The associated meshes used are shown in Figure 6 and Table I contains results obtained by the modified program.

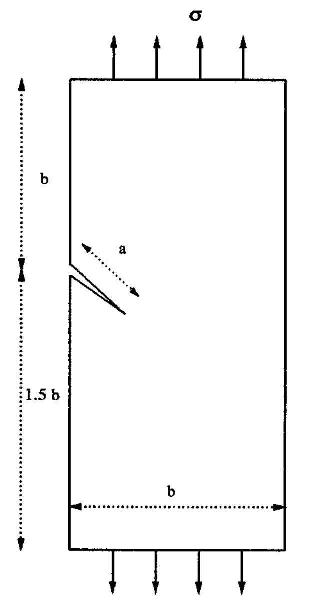

Figure 7. A mixed mode case for the separation of stress intensity factors

For cases A and B, the only difference is the use of the degenerated triangle-like elements, with four additional nodes as well as four additional elements in the latter. This causes a dramatic increase in accuracy, especially for K_{II} and the angle change, θ. Increasing the crack length for case C, with almost the same number of nodes and elements, with respect to case B; yields better results for K_I but, deteriorates in values for K_{II} and the angle change, θ. It is noteworthy that both cases utilize degenerated triangle-like elements.

7.2. Mixed mode condition

In this section the first example has been selected to demonstrate the separation of stress intensity factors while the second one represents a crack path determination.

7.2.1. A slant edge crack in a rectangular sheet. This example was chosen from a set of test cases solved by Rooke and Cartwright.[8] The applied load is a uniform uniaxial tensile stress as shown in Figure 7.

Similar to the previous example, various meshes have been used for the analyses, see Figure 8. Since in a mixed mode, the direction of the crack changes regularly; the program yields this angle change as well. The *Load Factor*, i.e. the inverse of the equivalent stress intensity factor, which indicates the change in the applied load; is also mentioned in Table II.

In this example, for the slant edge crack, case D uses a rather fine mesh which yields an accurate value for K_I but, some 15 per cent error in the computation of K_{II}. Meanwhile, case E utilizes a relatively coarse mesh with degenerated triangle-like elements which gives acceptable results, i.e. approximately 5 per cent error. In case F, the crack is wider and the numbers of nodes and

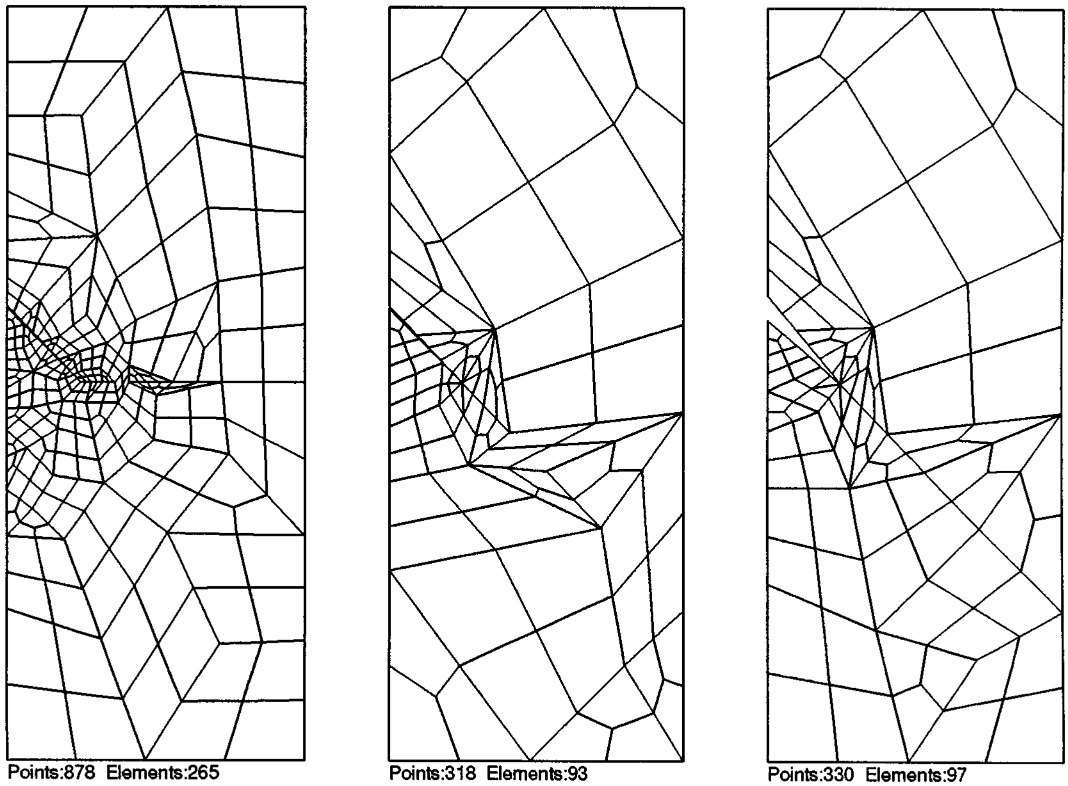

Figure 8. Various meshes used for mixed mode example series, cases D, E and F, respectively

elements are almost the same as case E. The value obtained for K_{I} has a 0.5 per cent error while K_{II} is, more or less, accurate. As expected, for this case the angle change lies between those of cases D and F.

7.2.2. Crack propagation under simulated cutter loading. It is well known within the finite element community that the stability and accuracy of a dynamic analysis is controlled by the time step size. In crack propagation problems, the stability and accuracy of the solution is controlled, in a similar way, by the crack length increment, Δa. Indeed, when a smaller value of Δa is taken the trajectory will be determined more accurately. In the mixed mode case, the crack trajectory is a curve because of the presence of K_{II}. For modelling purposes the curved path is to be evaluated via short straight lines. So, it is obvious that because of computational restrictions, an optimum choice should be considered such that when a reasonable number of steps are performed, the accuracy is acceptable. In other words, to prevent divergence or oscillation, the crack length increments must be reasonable. Unfortunately, it is difficult to put any strict restriction on Δa, because it depends on may factors. Usually, the user can find a suitable value after several attempts at a solution. Generally speaking, it seems that the more important factors are initial crack length, crack tip region element size, load level, ratio of $K_{\mathrm{II}}/K_{\mathrm{I}}$, global geometry of body, crack opening displacement and finally the user's experience and judgement.

Table II. The results obtained for the slant edge crack case

Case	No. of nodes	Crack length	Program		Angle change	Reference[9]	
			K_{I}	K_{II}		K_{I}	K_{II}
D	878	1·75	2219	927	− 36·3	2216	1108
E	318 ×	1·75	2122	1165	− 42·2	2216	1108
F	330 ×	1·75	2204	1108	− 40·3	2216	1108

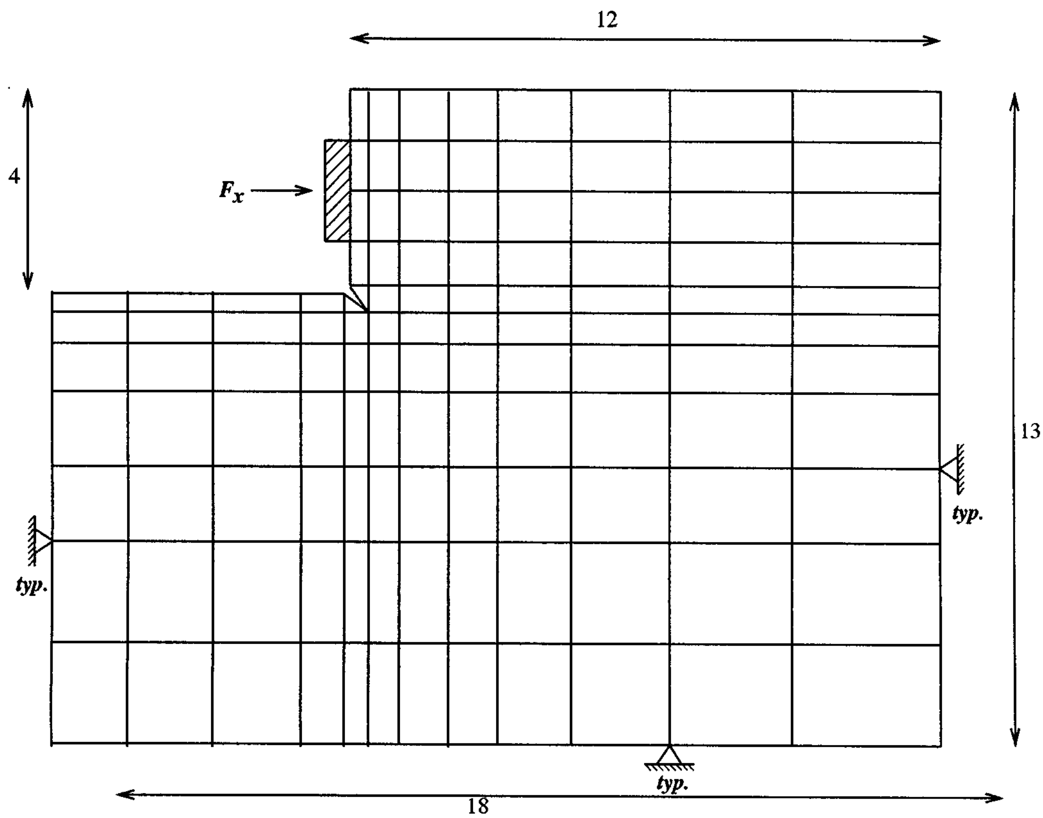

Figure 9. Initial crack direction and domain configuration

The last example simulated is one which presents the crack trajectory for the successive crack propagation in a rock mass, as shown in Figure 9. The material properties are assumed as

$$E = 5 \times 10^6 \text{ psi}$$

$$\nu = 0.3$$

$$K_{\mathrm{I_c}} = 1000 \text{ psi in}^{1/2}$$

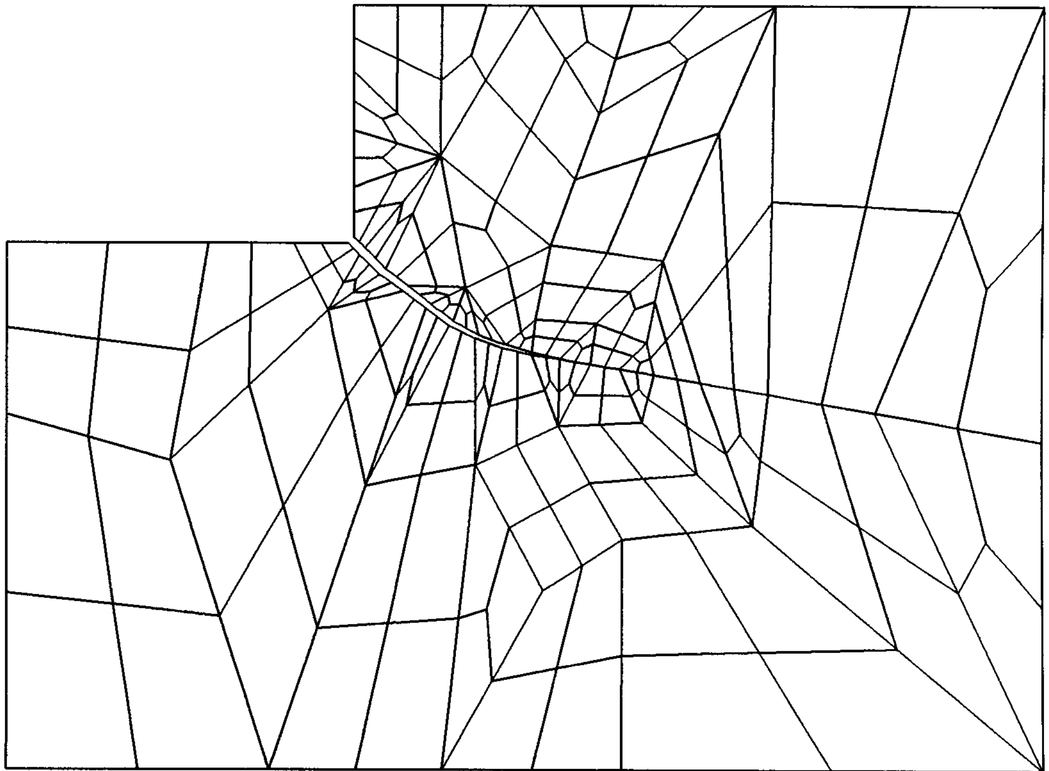

Figure 10. The final crack path configuration

Unfortunately, the initial crack length as well as successive numerical results have not been accurately stated by Ingraffea.[9] Hence, a comparison of the results is not easy. The final crack path, along with the associated mesh used, have been depicted in the last figure. The results, after each increment of the crack length, have been outlines in Table III. It should be mentioned that for all steps, degenerated triangle-like elements have been adopted in the vicinity of the crack tip.

For this case, because of the lack of sufficient data, e.g. initial crack length; it is difficult to compare the results obtained. Also the mesh generator was not robust. Therefore, in some increments of the crack length, unexpected results were encountered, especially in the few last increments as the crack path consisted of many segments. The smaller the crack length increment, the more accurate the results obtained. However, the greater the number of crack segments, the greater the number of elements and hence, a greater band width for the stiffness matrix. Generally speaking, the following guidelines are recommended.

(a) Create four or more rather equal elements at the crack tip with preferably similar angles at the crack tip before degenerating.
(b) A wider crack configuration, especially adjacent to the crack tip node.
(c) Choose a reasonable number of increments for the analyses.

Table III. Sequential results for the last example

Step	Crack length (in)	Applied load (lb)	Stress intensity factors		Angle change	Load factor
			K_I	K_{II}		
1	0·2	1000	331·0	2·1	− 0·73	3·02
2	0·7	3021	881·4	56·0	− 7·21	1·13
3	1·2	3407	908·5	3·7	− 0·47	1·10
4	1·7	3751	918·8	2·3	− 0·30	1·09
5	2·2	4081	925·0	77·4	− 9·64	1·09
6	2·7	4460	936·1	51·4	− 6·25	1·06
7	3·2	4745	943·5	59·7	− 7·19	1·05
8	3·7	4998	900·3	18·6	− 2·37	1·11

(d) Construct crack tip elements along the crack axis, which have a length not greater than 20 per cent of the crack length increment.

Finally, we can conclude that for this example the results are in general agreement with those presented by Ingraffea.[9]

8. CONCLUSIONS

This paper considers the maximum circumferential tensile stress criterion for crack propagation in mixed mode rock fracture mechanics. The displacement substitution method is used in order to compute the separated stress intensity factors for each mode simultaneously. The accuracy of the method for prediction of crack trajectory in mixed mode cases depends on crack length increments for each load change. A rather coarse mesh can be used for each step. The only constraint is to choose the same length for the two elements along the crack front.

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